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COMMENT

A non-Abelian Aharonov–Bohm effect in the framework of Feynman pseudo-classical path integrals

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Abstract. We analyse the non-Abelian Aharonov–Bohm effect in the Feynman path integral framework generalised to pseudo-classical mechanics.

The non-Abelian generalisation of the Aharonov–Bohm effect [1] was proposed by Wu and Yang [2] in 1975. In a recent comment [3] we analysed this non-Abelian effect by exploiting the existence of internal degrees of freedom associated with the particle's colour charge.

Our aim in this comment is to generalise the classical analysis implemented in [3] to the quantum level by using the Feynman path integral framework generalised to pseudo-classical mechanics.

Let us start our analysis by recalling some basic facts of pseudo-classical mechanics. In this framework the motion of a colour (spinless) massive charged particle is characterised by the usual (relativistic) vector position $x^\mu(\xi)$ added to a set of Grassman complex variables $\{\theta_l(\xi), \theta_l^*(\xi)\}$, where $\xi_i \leq \xi \leq \xi_f$ denotes a parameter describing the evolution of the system and $l = 1, \ldots, N$ is the gauge group number of generators $\{\lambda_i\}$ [5]. The pseudo-classical Lagrangian of a spinless colour particle in the presence of an external Yang-Mills gauge field $A_\mu(X) = A_\mu^i(X)\lambda_i$ is given by [4]

$$\mathcal{L}(X^\mu(\xi), \theta_l(\xi), \theta_l^*(\xi), A_\mu^i(X))$$

$$= -mc(\dot{X}^2(\xi))^{1/2} + \tfrac{1}{2}i \sum_{l=1}^{N} (\theta_l^* \dot{\theta}_l - \dot{\theta}_l^* \theta_l)(\xi)$$

$$- g(\theta_l^*(\lambda_i)_{lk}\theta_k)(\xi)A_\mu^i(X(\xi))\dot{X}^\mu(\xi). \tag{1}$$

In the Feynman path integral formalism for quantisation, the quantum propagation of the above physical system is given by the 'continuous' sum of all trajectories connecting the initial and final states characterised, respectively, by the initial vector position $X_\mu(\xi_i) = X_\mu$ and the colour degree $\theta_j(\xi_i) = \beta_j$ and $X_\mu(\xi_f) = Y_\mu$ with $\theta_k^*(\xi_f) = \eta_k^*$. We propose for this continuous sum the following path integral

$$G(|X_\mu, \beta_f>; |y_\mu, \eta_k^*)) = \int \left(\prod_{\substack{\xi_i < \xi < \xi_f \\ X_\mu(\xi_i) = X_\mu \\ X_\mu(\xi_f) = Y_\mu}} dX_\mu(\xi) \right) \left(\prod_{l=1}^{N} \prod_{\xi_i < \xi < \xi_f} d\theta_l(\xi)\, d\theta_l^*(\xi) \right)$$

$$\times (\theta_j(\xi_i)\theta_k^*(\xi_f)) \exp\left(\frac{i}{\hbar} \int_{\xi_i}^{\xi_f} \mathcal{L}(X^\mu(\xi), \theta_l(\xi), \theta_l^*(\xi), A_\mu^i(X))\, d\xi \right) \tag{2}$$

where $\mathcal{L}(X^\mu(\xi), \theta_l(\xi), \theta_l^*(\xi), A_\mu^i(X))$ is given by (1).

We note the similarity of the Grassmanian degrees propagation with the usual quantum field fermion propagation and (2) possesses group matrix indices. The remarkable feature of the proposed Grassmanian path measure is that we can evaluate it explicitly and it leads to the well known Wu-Yang factor as we show below, a result which generalises the Feynman result [6] on the Abelian case. This feature stems from the fact that the Grassmanian path integrals are of Gaussian type. So, we have to evaluate the expression:

$$
I_{jk}(X_\mu(\xi), A_\mu^i(X)) = \int \left(\prod_{i=1}^{N} \prod_{\xi_i < \xi < \xi_f} d\theta_l(\xi)\, d\theta_l^*(\xi) \right) (\theta_j(\xi_i) \cdot \theta_k^*(\xi_f))
$$
$$
\times \exp\!\left(\frac{i}{\hbar} \int_{\xi_i}^{\xi_f} \tfrac{1}{2} i \sum_{l=1}^{N} (\theta_l^* \dot\theta_l - \dot\theta_l^* \theta_l)(\xi) - g(\theta_l^*(\lambda_i)_{lk}\theta_k)(\xi) A_\mu^i(X(\xi))\dot X^\mu(\xi) \right).
$$
(3)

By noting the useful identity $(\theta_l^*\dot\theta_l - \dot\theta_l^*\theta_l)\,(\xi) = -(d/d\xi)(\theta_l\theta_l^*)\,(\xi) - 2(\dot\theta_l^*\theta_l)\,(\xi)$, and the result that the 'unidimensional' Grassmanian Green function is given by the 'non-relativistic' propagator $(d/d\tau)^{-1}\,(\Sigma_1, \Sigma_2) = \theta(\Sigma_1 - \Sigma_2)$, we can evaluate exactly the expression (3) ([7] see (2.20)):

$$
I_{jk}(X_\mu(\xi); A_\mu^i(X)) = \exp\{(1/2\hbar)[(\eta_k\eta_k^*) - (\beta_j\beta_j^*)]\}
$$
$$
\times \left[\mathbb{P}\!\left(e^{ig} \int_{\xi_i}^{\xi_f} A_\mu^i(X_\mu(\xi))\dot X^\mu(\xi)\lambda_i \right) \right]_{jk}.
$$
(4)

So we obtain the expression for the proposed propagator:

$$
G(|X_\mu, \beta_j\rangle; |Y_\mu, \eta_k^*\rangle)
$$
$$
= \int_{\substack{\xi_i < \xi < \xi_f \\ X_\mu(\xi_i)=X_\mu \\ X_\mu(\xi_f)=Y_\mu}} \prod dX_\mu(\xi) \exp\!\left[\frac{i}{\hbar}\left(\int_{\xi_i}^{\xi_f} -mc(\dot X^2(\xi))^{1/2}\, d\xi \right) \right] I_{jk}(X_\mu(\xi), A_\mu^i(X)).
$$
(5)

Now let us analyse the following 'Gedanken' experiment (see [3]): a quantum charged $U(N)$ spinless particle propagates in a region R where there is a non-zero Yang-Mills gauge field with the property that its associated field strength vanishes, with the initial and final states $|X_\mu, \beta_j\rangle; |Y_\mu, \eta_k^*\rangle$. We remark that this region can have an arbitrary topology. The associated quantum probability amplitude is given by (5), i.e. $G(|X_\mu, \beta_j\rangle; |Y_\mu, \eta_k^*\rangle)$.

Now we observe the following fundamental fact: the only system trajectory that contributes to the Wu-Yang factor in (5) comes from the classical trajectory, due to the vanishing field strength condition, as we can see by considering the functional Taylor expansion of (4) around the classical trajectory $\{X_{cl}^\mu(\xi); \xi_i \leq \xi \leq \xi_f\}$:

$$
[I(X_{cl}^\mu(\xi) + X_q^\mu(\xi); A_\mu^i(X))] - [I(X_{cl}^\mu(\xi); A_\mu^i(X))]
$$
$$
= \sum_{N=1} \frac{1}{N!} \int_{\xi_i}^{\xi_f} d\eta_1(X_q^{\alpha_1}(\eta_1))m \ldots
$$
$$
\times \int_{\xi_i}^{\xi_f} d\eta_N(X_q^{\alpha_N}(\eta_N)) \frac{\delta^{(N)} I[Z_\mu(\xi)]}{\delta X_q^{\alpha_1}(\eta_1) \ldots \delta X_q^{\alpha_N}(\eta_N)}\bigg|_{Z_\mu(\xi)=X_{cl}^\mu(\xi)}.
$$
(6)

Since all these functional derivatives (6) are multiplicatively proportional to the field strength (see for instance (20) and (21) of [8]); we achieve the above quoted result.

Now we consider the reversal quantum propagation, characterised by the initial and final states $|Y_\mu, \eta_k\rangle$ and $|X_\mu; \beta_j^*\rangle$: $G(|Y_\mu, \eta_\mu\rangle; |X_\mu; \beta_j^*\rangle)$. As usual, we would expect that the probability amplitude for the 'closed circuit' $(|X_\mu, \eta_k\rangle; |X_\mu, \beta_j^*\rangle)$ given by

$$\sum_{\{K\}} G(|X_\mu, \beta_j\rangle; |Y_\mu, \eta_k^*\rangle) G(|Y_\mu, \eta_k\rangle; |X_\mu, \beta_j^*\rangle)$$

should be a phase independent of the particular system trajectory connecting the above initial and final states. But due to the colour degrees (see (5) and (6)) we find that this phase is given by the Wu-Yang factor defined solely by the classical closed system trajectory.

So we have shown that the outcome of the proposed 'Gedanken' experiment, analysed on the quantum level, depends fundamentally on the (matrix elements) of the Wu-Yang non-integrable factor as obtained in our previous study [3] and predicted earlier by Wu and Yang.

Note added. After this comment was completed, we received the Preprint CPT-85/P-1769 by P A Horváthy, where physical applications of the above-mentioned non-Abelian Aharonov-Bohm effect is made.

References

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